XV. *On the mathematical theory of suspension bridges, with tables for facilitating their construction.* By Davies Gilbert, *Esq. V.P.R.S. &c. Communicated March* 9, 1826.

Read March 9, 1826.

My attention was first directed to a consideration of suspension bridges, and of the catenary curve on which their theory depends, when the plan for making such a communication across the Menai Straits was submitted to the Commissioners appointed by Parliament to improve the communication by roads and bridges through Wales. It then appeared to me, that the proposed depth of curvature, was not sufficient for ensuring such a degree of strength and permanence as would be consistent with the due execution of a great national work. This opinion I advanced as a Member of the Commission. But wishing to take on myself the full responsibility for such increased expense, as must of necessity be occasioned by enlarging the curvature, I also printed some approximations, hastily deduced, in the Quarterly Journal of Science; and derived from them a confirmation of the opinion that had been given. The interval between the points of support and the road-way of the Menai Bridge has in consequence been augmented to fifty feet; and it now possesses that full measure of strength, which experience has established as requisite and sufficient for works of iron not perfectly at rest.

Since bridges of suspension are obviously adapted to very general use, I have flattered myself with the hope of doing something serviceable to the public, by expanding into tables the formulæ from which my approximations were derived; adding to them other formulæ and tables for the catenary of equal strength. A curve not merely of speculative curiosity, but of practical use, where a wide horizontal extent may chance to be combined with natural facilities for obtaining a correspondent height for the attachments.

Both the ordinary catenaries, and these of equal strength, like circles, parabolas, logarithmic curves, &c. have the property of being each identical with themselves in every respect but size: and as the radius, the parameter, and the subtangent give the respective magnitudes of these curves, so are the catenaries determined in magnitude by the tension (expressed in measures of the chain), which takes place at the middle point, or apex of the curve, where it is a minimum. Consequently, when this tension is determined or given, all the other relations may be expressed in the same manner as sines, cosines, &c. in the circle.

I assume that the first principles of the catenary curve are known; they will, consequently, be noted with no other view, than to derive from them ulterior properties.

For the ordinary catenary:

Let $a =$ the tension at the apex, estimated in measures of the chain;

$x =$ the absciss, the versed sine, or depth of curvature;

$y =$ the ordinate, or semi-transverse length;

$z =$ the length of the curve.

Then since the tension, (a,) acts horizontally at the apex (A,) since the weight of the chain (z) acts at right angles to the former, and the force of suspension at (P) acts in the direction of the tangent. These forces must be represented in direction and in magnitude by the incremental triangle Prp; and

As $\dot{x} : \dot{y} :: z : a$; as $\dot{x}^2 : \dot{y}^2 :: z^2 : a^2$;

as $\dot{x}^2 + \dot{y}^2 : \dot{x}^2 :: a^2 + z^2 : z^2$;

But $\dot{x}^2 + \dot{y}^2 = \dot{z}^2$ universally.

Therefore, $\dot{z}^2 : \dot{x}^2 :: a^2 + z^2 : z^2$; and $\dot{x} = \frac{z\dot{z}}{\sqrt{a^2 + z^2}}$;

consequently, $x = \sqrt{a^2 + z^2} - a$.

$$\text{Equation A} \begin{cases} \text{No. 1.}\ x = \sqrt{a^2 + z^2} - a \\ \text{No. 2.}\ z = \sqrt{2ax + x^2} \\ \text{No. 3.}\ a = \dfrac{z^2 - x^2}{2x} \end{cases}$$

Again; $\dot{x} : \dot{y} :: z : a \because \dot{y} = \frac{a\dot{x}}{z}$, substituting from Eq. A. (No. 2.

$\dot{y} = \frac{a\dot{x}}{\sqrt{2ax + x^2}}$; and

Equ. B. No. 1.) $y = a \times$ natural log. of $\frac{a + x + \sqrt{2ax + x^2}}{a} =$ $a \times$ nat. log. $\frac{a + x + z}{a}$; or by substituting its value for a from Equ. A. No. 3, and dividing by $z + x$.

Equ. B. No. 2.) $y = a \times$ nat. log. $\frac{z + x}{z - x}$;

or, if $\frac{z\dot{z}}{\sqrt{a^2 + z^2}}$ be substituted for $\dot{x}$ in

$$\dot{y} = \frac{a\dot{x}}{z}. \quad \dot{y} = a \times \frac{\dot{z}}{\sqrt{a^2 + z^2}}$$

and

Equ. B. No. 3.) $y = a \times$ nat. log. $\frac{\sqrt{a^2 + z^2} + z}{a}$.

To find x when a and y are given:

Let N = the number of which $\frac{y}{a}$ (Equ. B. No. 1.) is the natural logarithm.

Then $a\text{N} = a + x + \sqrt{2ax + x^2}$, and $\sqrt{2ax + x^2} = a\text{N} - a - x$, make $a\text{N} - a = \text{M}$. Then $2ax + x^2 = \text{M}^2 - 2\text{M}x + x^2$, and

Equ. C.) $x = \frac{\text{M}^2}{2\text{M} + 2a}$,

x being known.

z is found from Equ. A. No. 2. and

T, the tension at P, being obviously equal to $\sqrt{a^2 + z^2}$, is equal (Eq. A. No. 2.) to $\sqrt{a^2 + 2ax + x^2} = a + x$.

The angle of suspension is derived from the common analogy of the incremental triangle, and of the forces corresponding with it.

Tables I. and II. are constructed from these theorems, and their use will be best explained by an example.

Let the span proposed for a suspension bridge be 800 feet, and let the adjunct weight of suspension rods, road-way, &c. be taken at one-half of the weight of the chains; then, if the full tenacity of iron is represented by the modulus of 14800 feet, the virtual modulus for the whole weight must be reduced in the proportion of 2 + 1 : 2, or to 9867 feet; and let it be determined to load the chains at the point of their greatest strain, that is at the points of suspensions, with one-sixth part of the weight they are theoretically capable of sustaining.

Then, since the semi-span is 400 feet, and y in Table I. is taken at an hundred measures, each of these measures must be four feet, and the weight expressed in the same measures to be sustained at the points of suspension will be $9867 \div 6 \times 4 = 411,125$. Now it appears from Table I. where y is uniformly an hundred, that when T = 412

$a = 400$ measures or 1600 feet.

$x = 12.565$ - - 50.260

$z = 101.045$ - 404.180

$\angle$ the angle of suspension 75° 49′.

Having now determined a, the modulus, latus rectum, or parameter of the curve. In Table II. will be found all the respective quantities for each measure of y. But as a is in this table taken at an hundred measures, and it has been found to be 400 of the former, each measure here must be 4 times 4, or 16 feet; consequently, each gradation of y will also be 16 feet, and the whole semi-span $\frac{400}{16}$ or 25 measures. And since z will be given in the Table for each measure of y, the adjunct weights may readily be adapted to a strict preservation of the catenary form.

At 21 measures of y. $z = 21.1547$
20 measures of y. $z = 20.1335$

$1.0212 \times 16 = 16.3392$ feet.

Consequently while the ordinate extends one measure, or 16 feet from the 20th to the 21st measure, the length of the curve will increase 16 feet and $\frac{1}{3}$ very nearly, and the adjunct weight should be increased in the same proportion.

At 21 the length of x is 2,2131 measures, or multiplied by $16 = 35{,}4096$ feet, the length of the suspension rods to the level of the apex.

It appears from Table I. that the tension T for a given half span of 100 measures is very nearly at its minimum when $x = 65{,}85$ measures, almost one-third part of the whole span. In the example taken above $65.85 \times 4 = 263{,}4$ feet, an height not to be attained in practice, nor strictly upplicable if it could be reached, because of the great length of suspension. If the span and height ($2\,y$ and x) were given, the other quantities would be found in a similar manner.

In the catenary of equal strength

$a . x . y . z$ remain as before; but another symbol must now be introduced, $\zeta =$ the mass of the chain. Then will the forces be represented as in the ordinary curve by the incremental triangle Prp. But now $\dot{x} : \dot{y} :: \zeta : a$. And by a repetition of the former steps $\dot{x} = \frac{\zeta \dot{z}}{\sqrt{a^2 + \zeta^2}}$.

But on the principle of equal strength,

As $a : \sqrt{a^2 + \zeta^2} :: \dot{z} : \dot{\zeta}$

therefore $\dot{z} = a \times \frac{\dot{\zeta}}{\sqrt{a^2 + \zeta^2}}$ and

Equ. D.) $z = a \times$ nat. log. $\frac{\sqrt{a^2 + \zeta^2} + \zeta}{a}$;

and by substituting $a \times \frac{\dot{\zeta}}{\sqrt{a^2 + \zeta^2}}$ for $\dot{z}$ in the equation $\dot{x} = \frac{\zeta \dot{z}}{\sqrt{a^2 + \zeta^2}}$,

$\dot{x} = a \times \frac{\zeta \dot{\zeta}}{a^2 + \zeta^2}$; consequently,

Equ. E.) $x = \frac{a}{2} \times$ nat. log. $\frac{a^2 + \zeta^2}{a^2}$.

Again, from the first analogy, $\dot{y} = \frac{a \dot{z}}{\zeta}$,

substitute for $\dot{x}$ its equal $a \times \frac{\zeta \dot{\zeta}}{a^2 + \zeta^2}$, and

$\dot{y} = a^2 \times \frac{\dot{\zeta}}{a^2 + \zeta^2}$; therefore,

Equ. F.) $y =$ the cir. arc. of which ζ is the tangent to radius a.

a and y being given to find ζ. Multiply $\frac{y}{a}$ by $57^\circ,29578$ (the tab. log. 1.7581226) and reduce the decimals of a degree into minutes and seconds; then will the tangent of that arc, multiplied by a, be equal to ζ.

And when ζ has been determined, the other columns of Tables III. and IV. are constructed from the above theorems, in a manner perfectly similar to that used in calculating of Tables I. and II.; and they may be illustrated by the same example; observing that a, now represents the uniform tension on each given magnitude of iron throughout the chains, and that the column T has the whole pull which any building or support may have to sustain in the direction of the tangent.

In Table III. y being, as before, an hundred measures of four feet each, a must be sought $=411,125$, and by proportioning between 420 and 400

$x=$ 12.2904	measures or	49.1616	Feet.
$z=$ 101.0020		404.0080	
$\zeta=$ 102.0235		408.0940	
T $=$ 423.6019		1694.4076	

$< . 76^{\circ}\ 3'\ 17''$.

a, or the modulus of this curve being fixed at 411,125 measures of 4 feet each, or at 1644,5 feet; and a in Table IV. being taken at 100 measures, each one will be 16,445 feet, and all the quantities are given for each gradation of y.

Thus at 21 measures of y . $z=21.1564$ $\zeta=21.3142$
20 measures of y . $z=20.1347$ $\zeta=20.2710$
1,0217 1.04332

$1.0217 \times 16.445=16.8019$ feet the increase of z;
$1.0432 \times 16.445=17.1410$ feet the increase of material in ζ: consequently $\frac{1.0432}{1.0217}=1.021$, the quantity of matter in this part of the chain to maintain uniform strength, that at the apex being unity, and the adjunct matter should be in the proportion of 1 to 1,04332.

Moreover x the versed sine, or the length of the suspension rods to the level of the apex will be at
21 measures of y . $x=2.2214$ measures $\times$ 16,445 $=36,531$ ft.
20 measures of y . $x=2.0135$ measures $\times$ 16.445 $=33.112$ ft.

Assuming in the ordinary catenary that $x = 65.85$ measures, is the height of the attachment to give a maximum extent of span with any virtual tenacity of material, a will be 85 measures, and $a + x = 85 + 65.85$, or 150,85 measures equal the given virtual tenacity. This taken as before at $\frac{2}{3}$ of $\frac{1}{6}$ of 14800 feet, will give 10,875 feet for each measure, and the whole span at $2y = 2175$ feet. Chains merely supporting themselves, and at the utmost of their tenacity will extend nine times further, or to 19575 feet.

In the catenary of equal strength, the semi-span being equal to the circular arc of which ζ is the tangent to radius a, it is obvious that $a \times$ semi-cir. arc must be the limit of the span. Therefore if $a = \frac{2}{3}$ of $\frac{1}{6}$ of 14800 feet, or 1644,44 $a \times \frac{c}{2} = 5154$ feet.

And if the chains merely sustain themselves at their utmost tenacity, 5154×9 will give 46385 feet, equal to 8,785 miles, or somewhat more than 8 miles and three-quarters.

But this case is purely hypothetical, for the purpose of ascertaining a limit, since ζ, the mass or weight of the chain must be infinite, and consequently its length: the figure approaching indefinitely near to that of a chain sustaining itself from an infinite height, which figure is identical with that of a building, capable so far as pressure and the strength of materials are alone concerned, of being carried to any elevation whatever. This figure is readily determined:

Let $a =$ the section of such a building at its base,
$y =$ the section at any height,
$x =$ that height;

Then, since the section and the superincumbent pressure must always be in the same proportion to each other, $\dot{x}$ and $\frac{\dot{y}}{y}$ are in a constant ratio. Let then $\frac{\dot{x}}{m} = -\frac{\dot{y}}{y}$ where m is the modulus of pressure in the given material; but when $x = o, y = a$, therefore $\frac{x}{m}$ = the nat. log. $\frac{a}{y}$; or $\frac{x}{A.m}$ = the tab. log. $\frac{a}{y}$. A = 2,3025851; but if ε and γ the homologous sides or diameters of these sections; then, $\frac{x}{2.A.m}$ = tab. log. $\frac{\varepsilon}{\gamma}$.

Finally, I would notice a correction of frequent use in practical surveying, to be deduced from the properties of the catenary curve.

When the measuring chain is extended over ground uneven, intersected by ditches, or made soft by water, it cannot be laid flat, but must be elevated at both its extremities, while the middle just touches the surface: thus giving the measurement too great by the difference between the whole perifery and the double ordinate.

Let z = the half length of the chain.

x = the elevation at each end equal to the depths of curvature.

Then Equ. B. No. 2. $y = a \times$ nat. log. $\frac{z+x}{z-x}$,

And Equ. A. No. 3. $a = \frac{z^2 - x^2}{2x}$; therefore

$$y = \frac{z^2 - x^2}{2x} \times \text{nat. log.} \frac{z+x}{z-x}.$$

But when x is very small in comparison of z, the nat. log. of $\frac{z+x}{z-x}$ becomes $\frac{2x}{z}$, and

$$y = \frac{z^2 - x^2}{2x} \times \frac{2x}{z} = z - \frac{x^2}{z};$$

$\frac{x^2}{z}$ is therefore the difference between half the chain and the ordinate. If x be expressed in parts of the whole chain, $4\,x^2$ will be the correction for the difference between the perifery and double ordinate.

If x (the elevation at each end) be one link of the common measuring chain, $4\,x^2 = \frac{1}{25}$ of a link, $\frac{1}{25}$ of $\frac{66}{100}$ of a foot $= 0.3168$ of an inch, varying as the squares of x.

If half the chain were considered as a straight line, and the hypothenuse of a right angled triangle, the horizontal distance would be $z - \frac{x^2}{2\,z}$, giving but one half of the true difference, 0,1584 parts of an inch.

And if the chain were supposed to be in the arc of a circle, $z = y + \frac{y^3}{6\,a\,z}$, &c. And $y = \sqrt{2\,a\,x - x^2}$ (when x is very small in comparison with a) $= \sqrt{2\,a\,x}$. Therefore $a = \frac{y^2}{2\,x}$.

And since y is also small in comparison of a, the second term of the series $\left(\frac{y^3}{6\,a^2}\right)$ will be the difference between the ordinate and the arc. Then substituting $\frac{y^4}{4\,x^2}$ for a^2, $\frac{y^3}{6\,a^2} = \frac{2\,x^2}{3\,y}$; or if x be expressed in parts of the whole chain, $= \frac{8}{3}\,x^2$ will be the whole correction, $= 0.2112$ parts of an inch, or two-thirds of the true difference.

Formulæ might readily be constructed for different elevations of the extremities of the chain, but they would prove much too complicated for practical use.

One further observation may be applicable to suspension bridges, wholly unconnected with the preceding investigations.

In the event of their wanting stability to counteract and restrain undulatory motion, the ballustrades may be carried to any required height, and rendered inflexible by diagonal braces; and if further means were required for imparting stability, such braces might be adjusted with screws to the suspension rods themselves, after these rods had acquired their exact positions, on the completion of the work.

$y = 100.$ **Table I.— Ordinary Catenary.**

a.	N.	*x.*	*z.*	T.	Angle.
2000	1.051271	2.500511	100.041474	2002.500511	87° 8′ 11″
1950	1.052619	2.564593	100.042440	1952.564593	87 3 46
1900	1.054041	2.632163	100.045727	1902.632163	86 59 8
1850	1.055541	2.703298	100.047540	1852.703298	86 54 15
1800	1.057127	2.778421	100.050163	1802.778421	86 49 6
1750	1.058807	2.857914	100.054318	1752.857914	86 43 40
1700	1.060588	2.942018	100.057566	1702.942018	86 37 53
1650	1.062480	3.031204	100.060788	1653.031204	86 31 46
1600	1.064494	3.125974	100.064421	1603.125974	86 25 16
1550	1.066642	3.226852	100.068245	1553.226852	86 18 21
1500	1.068939	3.334558	100.073939	1503.334558	86 10 59
1450	1.071399	3.449618	100.078929	1453.449618	86 3 6
1400	1.074041	3.572907	100.084490	1403.572907	85 54 39
1350	1.076886	3.705344	100.090750	1353.705344	85 45 35
1300	1.079958	3.847958	100.097440	1303.847958	85 35 45
1250	1.083286	4.002035	100.105463	1254.002035	85 25 16
1200	1.086903	4.168981	100.114680	1204.168981	85 13 51
1150	1.090849	4.350543	100.125801	1154.350543	85 1 26
1100	1.095169	4.548545	100.137346	1104.548545	84 47 54
1050	1.099920	4.765440	100.150553	1054.765440	84 33 5
1000	1.105170	5.004084	100.165906	1005.004084	84 16 48

The column in Table I. marked N (where the numbers equal $e^{\frac{y}{a}}$) is given as the medium conducting to all the subsequent calculations.

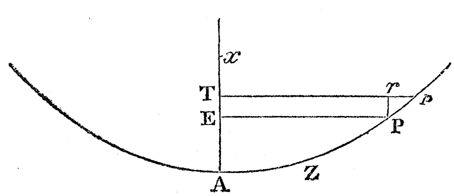

Table I. *continued.*—The Ordinary Catenary.

$y = 100.$

a.	N.	*x.*	*z.*	T.	Angle.
1000	1.105170	5.004084	100.165906	1005.004084	84° 16′ 48″
980	1.107428	5.106408	100.173025	985.106408	84 9 49
960	1.109785	5.213007	100.180582	965.213007	84 2 13
940	1.112247	5.324098	100.188974	945.324098	83 54 58
920	1.114822	5.440045	100.196191	925.440045	83 47 4
900	1.117519	5.561266	100.205825	905.561266	83 38 48
880	1.120344	5.687876	100.214837	885.687876	83 30 11
860	1.123309	5.820479	100.225255	865.820479	83 21 9
840	1.126423	5.959364	100.235949	845.959364	83 11 42
820	1.129698	6.105033	100.247321	826.105033	83 1 47
800	1.133148	6.258102	100.260296	806.258102	82 51 23
780	1.136785	6.418938	100.273356	786.418938	82 40 28
760	1.140627	6.588360	100.288153	766.588360	82 28 57
740	1.144691	6.767004	100.304328	746.767004	82 16 50
720	1.148996	6.955577	100.321527	726.955577	82 4 3
700	1.153564	7.154926	100.339869	707.154926	81 50 33
680	1.158422	7.366193	100.360765	687.366193	81 36 15
660	1.163595	7.590181	100.382517	667.590181	81 21 6
640	1.169118	7.828368	100.407143	647.828368	81 5 1
620	1.175025	8.081923	100.433570	628.081923	80 47 54
600	1.181360	8 352608	100 463404	608.352608	80 29 40
580	1.188169	8.642033	100.495985	588.642033	80 10 11
560	1.195508	8.952299	100.532176	568.952299	79 49 27
540	1.203419	9.283888	100.562366	549.283888	79 27 2
520	1.212043	9.645021	100.617335	529.645021	79 2 56
500	1.221402	10.033315	100.667683	510.033315	78 36 59
480	1.231625	10.454508	100.725490	490.454508	78 8 55
460	1.242830	10.912412	100.789382	470.912412	77 38 28
440	1.255172	11.412622	100.863052	451.412622	77 5 23
420	1.268829	11.961025	100.947150	431.961025	76 29 6
400	1.284025	12.565207	101.044792	412.565207	75 49 22
380	1.301032	13.233994	101.158163	393.233994	75 5 35
360	1.320192	13.978365	101.290757	373.978365	74 17 7
340	1.341941	14.812141	101.447796	354.812141	73 32 10
320	1.366837	15.752501	101.635337	335.752501	72 22 46
300	1.395612	16.821529	101.862069	316.821529	71 14 44
280	1.429239	18.047685	102.139232	298.047685	69 57 31
260	1.469049	19.468993	102.483745	279.468993	68 29 13
240	1.516896	21.126437	102.893226	261.126437	66 47 38
220	1.575420	23.118850	103.473548	243.118850	64 48 38
200	1.648721	25.525175	104.219022	225.525175	62 28 34
180	1.743908	28.559946	105.343499	208.559946	59 39 43
160	1.868245	32.280531	106.638654	192.280531	56 19 0
140	2.042722	37.258541	108.722538	177.258541	52 10 2
120	2.300975	44.134402	111.982596	164.134402	46 58 48
100	2.718281	54.308027	117.520071	154.308027	40 23 42
95	2.865180	57.674415	119.517684	152.674415	38 28 45
90	3.037731	61.511583	121.884206	151.511583	36 26 34
85	3.240907	65.852160	124.624934	150.852160	34 17 44
80	3.490342	71.073875	128.153485	151.073875	31 58 28
75	3.793667	77.147407	132.377616	152.147407	29 32 4
70	4.172733	84.433443	137.657866	154.433443	26 57 10

Table II.—The Ordinary Catenary.

$a = 100$.

N.	*y.*	*x.*	*z.*	T.	Angle.
					° ′ ″
1.010050	1	.004999	1.000000	100.004999	89 25 39
1.020201	2	.020000	2.000100	100.020000	88 51 15
1.030454	3	.045001	3.000398	100.045001	88 16 53
1.040810	4	.080007	4.000992	100.080007	87 42 31
1.051271	5	.125025	5.002074	100.125025	87 8 11
1.061836	6	.180050	6.003540	100.180050	86 33 51
1.072508	7	.245098	7.005701	100.245098	85 59 33
1.083287	8	.320170	8.008520	100.320170	85 25 16
1.094174	9	.405271	9.012128	100.405271	84 51 1
1.105170	10	.500408	10.016591	100.500408	84 16 48
1.116278	11	.605609	11.022190	100.605609	83 42 36
1.127496	12	.720855	12.028744	100.720855	83 8 37
1.138828	13	.846186	13.036613	100.846186	82 34 20
1.150273	14	.981591	14.045708	100.981591	82 0 14
1.161834	15	1.127107	15.056292	101.127107	81 26 15
1.173510	16	1.282710	16.068289	101.282710	80 52 17
1.185304	17	1.448471	17.081928	101.448471	80 18 22
1.197217	18	1.624373	18.097326	101.624373	79 44 31
1.209249	19	1.810427	19.114472	101.810427	79 10 43
1.221402	20	2.006663	20.133536	102.006663	78 36 59
1.233678	21	2.213114	21.154685	102.213114	78 3 19
1.246076	22	2.429763	22.177836	102.429763	77 29 43
1.258600	23	2.656680	23.203319	102.656680	76 56 11
1.271249	24	2.893847	24.231042	102.893847	76 22 45
1.284025	25	3.141302	25.261197	103.141302	75 49 22
1.296929	26	3.399061	26.293838	103.399061	75 16 5
1.309964	27	3.667187	27.329212	103.667187	74 42 53
1.323129	28	3.945662	28.367237	103.945662	74 9 46
1.336427	29	4.234542	29.408157	104.234542	73 36 44
1.349858	30	4.533833	30.451966	104.533833	73 3 48
1.363424	31	4.843577	31.498822	104.843577	72 30 58
1.377127	32	5.163822	32.548877	105.163822	71 58 13
1.390968	33	5.494589	33.602210	105.494589	71 25 35
1.404947	34	5.835881	34.658818	105.835881	70 53 3
1.419067	35	6.187768	35.718931	106.187768	70 20 36
1.433329	36	6.550276	36.782623	106.550276	69 48 18
1.447734	37	6.923431	37.849968	106.923431	69 16 6
1.462284	38	7.307284	38.921115	107.307284	68 44 0
1.476980	39	7.701863	39.996336	107.701863	68 12 1
1.491824	40	8.107217	41.075182	108.107217	67 40 10
1.506817	41	8.523379	42.158320	108.523379	67 8 25
1.521961	42	8.950402	43.245697	108.950402	66 36 48
1.537257	43	9.388315	44.337384	109.388315	66 5 19
1.552706	44	9.837146	45.433453	109.837146	65 33 57
1.568312	45	10.297011	46.534188	110.297011	65 2 43
1.584073	46	10.767851	47.639448	110.767851	64 31 46
1.599994	47	11.249817	48.749582	111.249817	64 0 39
1.616074	48	11.742877	49.864522	111.742877	63 29 49
1.632315	49	12.247092	50.984407	112.247092	63 59 7
1.648721	50	12.762587	52.109512	112.762587	62 28 34

Table II. *continued.*—The Ordinary Catenary.

$a = 100$.

N.	*y.*	*x.*	*z.*	T.	Angle.
					° ′ ″
1.665290	51	13.289300	53.239600	113.289300	61 58 9
1.682027	52	13.827388	54.375311	113.827388	61 27 53
1.698932	53	14.376853	55.516346	114.376853	60 57 45
1.716006	54	14.937727	56.662872	114.937727	60 27 46
1.733252	55	15.510107	57.815092	115.510107	59 57 56
1.750672	56	16.094061	58.973138	116.094061	59 28 14
1.768266	57	16.689588	60.137011	116.689588	58 58 42
1.786037	58	17.296790	61.306900	117.296790	58 29 19
1.803988	59	17.915770	62.483020	117.915770	58 0 5
1.822118	60	18.546493	63.665306	118.546493	57 31 1
1.840431	61	19.189099	64.854000	119.189099	57 2 5
1.858927	62	19.843586	66.049113	119.843586	56 33 20
1.877610	63	20.510098	67.250901	120.510098	56 4 43
1.896480	64	21.188633	68.459366	121.188633	55 36 16
1.915540	65	21.879300	69.674600	121.879300	55 7 59
1.934792	66	22.582171	70.897028	122.582171	54 39 52
1.954237	67	23.297283	72.126416	123.297283	54 11 54
1.973877	68	24.024709	73.362990	124.024709	53 44 6
1.993715	69	24.764560	74.606930	124.764560	53 16 28
2.013752	70	25.516873	75.858326	125.516873	52 48 59
2.033990	71	26.281725	77.117274	126.281725	52 21 41
2.054433	72	27.059265	78.384034	127.059265	51 54 33
2.075080	73	27.849426	79.658573	127.849426	51 27 34
2.095935	74	28.652451	80.941048	128.652451	51 0 46
2.117000	75	29.468327	82.231672	129.468327	50 34 8
2.138276	76	30.297123	83.530476	130.297123	50 7 40
2.159766	77	31.138956	84.837643	131.138956	49 41 22
2.181472	78	31.993903	86.153296	131.993903	49 15 14
2.203396	79	32.862044	87.477555	132.862044	48 49 16
2.225540	80	33.743457	88.810542	133.743457	48 23 29
2.247907	81	34.638263	90.152436	134.638263	47 57 52
2.270500	82	35.546581	91.503418	135.546581	47 32 25
2.293318	83	36.468371	92.863428	136.468371	47 7 8
2.316366	84	37.403837	94.232762	137.403837	46 42 2
2.339646	85	38.353056	95.611543	138.353056	46 17 6
2.363160	86	39.316110	96.999880	139.316110	45 52 20
2.386910	87	40.293084	98.397915	140.293084	45 27 45
2.410900	88	41.284143	99.805856	141.284143	45 3 20
2.435129	89	42.289243	101.223656	142.289243	44 39 5
2.459602	90	43.308592	102.651607	143.308592	44 15 1
2.484322	91	44.342313	104.089886	144.342313	43 51 7
2.509290	92	45.390455	105.538544	145.390455	43 27 23
2.533983	93	46.430931	106.967368	146.430931	43 4 18
2.559981	94	47.530444	108.467655	147.530444	42 40 26
2.585709	95	48.622506	109.948393	148.622506	42 17 13
2.611696	96	49.729447	111.440152	149.729447	41 54 10
2.637944	97	50.851184	112.943315	150.851184	41 31 18
2.664455	98	51.988313	114.457186	151.988313	41 8 36
2.691234	99	53.140537	115.982862	153.140537	40 46 4
2.718281	100	54.308027	117.520072	154.308027	40 23 42

Table III. The Catenary of equal strength.

$y = 100.$

a.	*x.*	*z.*	ζ.	T.	Angle.
					° ′ ″
1000	5.008288	100.166600	100.334300	1005.020800	84 16 13
980	5.110881	100.173640	100.348276	985.124220	84 9 12
960	5.217781	100.181250	100.363200	965.232000	84 1 54
940	5.329126	100.188850	100.378652	945.344276	83 54 16
920	5.445471	100.197071	100.395276	925.461672	83 46 19
900	5.566977	100.202654	100.413000	905.584230	83 38 1
880	5.694003	100.215533	100.432288	885.712432	83 29 20
860	5.827073	100.225792	100.452730	865.846882	83 20 15
840	5.966506	100.237329	100.475340	845.987772	83 10 44
820	6.112609	100.247806	100.497724	826.135404	83 0 45
800	6.266274	100.261054	100.523680	806.290880	82 50 16
780	6.427811	100.274596	100.551048	786.454344	82 39 15
760	6.598152	100.289657	100.580680	766.626896	82 27 40
740	6.777369	100.305695	100.613064	746.808518	82 15 25
720	6.966790	100.322732	100.647648	727.000675	82 2 32
700	7.167238	100.342923	100.685480	707.204050	81 48 53
680	7.379542	100.362168	100.726972	687.419752	81 34 26
660	7.604848	100.384645	100.772166	667.647826	81 19 7
640	7.844443	100.409125	100.821568	647.892736	81 2 51
620	8.099715	100.436355	100.876232	628.152876	80 45 31
600	8.370382	100.465969	100.936080	608.430840	80 27 2
580	8.663690	100.498855	101.002534	588.728710	80 7 17
560	8.976381	100.535447	101.076360	569.048704	79 46 7
540	9.312582	100.576282	101.158740	549.393354	79 23 23
520	9.675126	100.621836	101.250968	529.765704	78 58 53
500	10.067350	100.679481	101.362400	510.169400	78 32 27
480	10.552010	100.780247	101.472192	490.668864	78 3 48
460	10.956213	100.796941	101.605490	471.087748	77 32 39
440	11.462781	100.872044	101.757920	451.613404	76 58 41
420	12.018908	100.958305	101.933328	432.192558	76 21 29
400	12.630692	101.056700	102.136560	412.832200	75 40 33
380	13.312576	101.174410	102.373976	393.548520	74 55 19
360	14.071210	101.311236	102.653784	374.349852	74 5 4
340	14.922900	101.473699	102.986884	355.255222	73 8 53
320	15.886128	101.668413	103.387488	336.287040	72 5 42
300	16.984763	101.904940	103.875990	317.474760	70 54 5
280	18.250135	102.196102	104.480264	298.858028	69 32 14
260	19.729226	102.564124	105.241136	280.497074	67 57 47
240	21.465587	103.025715	106.219200	262.454784	66 7 36
220	23.555838	103.632647	107.507994	244.863168	63 57 23
200	26.116574	104.447443	109.260480	227.898480	61 21 7
180	29.336487	105.580330	111.739482	211.862484	58 10 8
160	33.525185	107.228464	115.437376	197.296208	54 11 24
140	39.241137	109.779803	121.380952	185.292618	49 4 28
120	47.626016	114.104417	132.093348	178.461912	42 15 12
100	61.562643	122.619114	155.740770	185.081570	32 42 15
95	66.748734	126.148321	166.629316	191.808059	29 41 19
90	73.141390	130.727676	181.797084	202.855068	26 20 16
85	81.313401	136.905055	204.267512	221.246959	22 35 35
80	92.332784	145.717467	240.765568	253.708616	18 22 48
75	108.536763	159.466590	309.878850	318.825817	13 36 20
70	136.763450	184.926359	488.855143	493.841432	8 8 56

Table IV.—The Catenary of equal strength.

$a = 100.$

y.	*x.*	*z.*	*ζ.*	T.	Angle.
					° ′ ″
1	.004999	.999990	1.00001	100.00500	89 25 37
2	.020003	2.000088	2.00022	100.020006	88 51 14
3	.045005	3.000431	3.00088	100.045016	88 16 52
4	.080021	4.001021	4.00208	100.080054	87 42 29
5	.125046	5.002067	5.00415	100.125125	87 8 6
6	.180107	6.003541	6.00714	100.180270	86 33 44
7	.245198	7.005697	7.01143	100.245499	85 59 21
8	.323389	8.008498	8.01706	100.320852	85 24 58
9	.405548	9.012161	9.02436	100.406373	84 50 46
10	.500828	10.016660	10.03343	100.502080	84 16 13
11	.606218	11.022229	11.04456	100.608062	83 41 50
12	.721234	12.028425	12.05789	100.723845	83 7 28
13	.847386	13.036754	13.07372	100.850992	82 33 5
14	.983205	14.045921	14.09215	100.988063	81 58 42
15	1.129248	15.056560	15.11351	101.135644	81 24 20
16	1.285490	16.068670	16.13791	101.293792	80 49 57
17	1.452011	17.082468	17.16567	101.462608	80 15 34
18	1.628815	18.097959	18.19691	101.642158	79 41 12
19	1.815961	19.115360	19.23197	101.832558	79 6 49
20	2.013470	20.134658	20.27097	102.033880	78 32 23
21	2.221395	21.156371	21.31424	102.246255	77 58 4
22	2.439770	22.179619	22.36191	102.469780	77 23 41
23	2.668651	23.205504	23.41433	102.704585	76 49 19
24	2.908061	24.233742	24.47164	102.950768	76 14 56
25	3.158106	25.264601	25.53424	103.208504	75 40 33
26	3.418774	26.297360	26.60212	103.477887	75 6 11
27	3.690164	27.334154	27.67581	103.759100	74 31 48
28	3.972311	28.373174	28.75540	104.052264	73 57 25
29	4.265294	29.415243	29.84128	104.357567	73 23 3
30	4.569158	30.460378	30.93360	104.675156	72 48 40
31	4.883983	31.508739	32.03269	105.005213	72 14 17
32	5.209839	32.560521	33.13891	105.347935	71 39 55
33	5.546782	33.615738	34.25243	105.703501	71 5 32
34	5.894915	34.674639	35.37366	106.072131	70 31 9
35	6.254281	35.737235	36.50280	106.454005	69 56 47
36	6.624997	36.803792	37.64030	106.849383	69 22 24
37	7.007106	37.874291	38.78626	107.258446	68 48 2
38	7.400749	38.948988	39.94126	107.681495	68 13 39
39	7.805967	40.027947	41.10545	108.118722	67 39 16
40	8.222888	41.111407	42.27931	108.570433	67 4 54
41	8.651589	42.199404	43.46308	109.036870	66 30 31
42	9.092196	43.292198	44.65724	109.518354	65 56 8
43	9.544771	44.389841	45.86509	110.015128	65 21 46
44	10.009478	45.492556	47.07804	110.527566	64 47 23
45	10.486371	46.600436	48.30547	111.042096	64 13 0
46	10.975622	47.713735	49.54487	111.600602	63 38 38
47	11.477312	48.832499	50.79655	112.161892	63 4 15
48	11.991595	49.957023	52.06108	112.740211	62 29 52
49	12.518572	51.088569	53.34078	113.335897	61 55 32
50	13.058418	52.223810	54.63024	113.949396	61 21 7

Table IV. *continued.*—The Catenary of equal strength.

$a = 100$.

y.	*x*.	*z*.	ζ.	T.	Angle.
					° ′ ″
51	13.611226	53.366417	55.93584	114.581052	60 46 44
52	14.177189	54.515494	57.25618	115.231377	60 12 22
53	14.756401	55.676950	58.59167	115.900748	59 37 59
54	15.349077	56.833577	59.94296	116.589191	59 3 36
55	15.955315	58.002974	61.31049	117.298661	58 29 14
56	66.575346	59.179619	62.69495	118.028208	57 54 51
57	17.209276	60.363609	64.09682	118.778802	57 20 29
58	17.857313	61.555215	65.51678	119.551032	56 46 6
59	18.519676	62.754711	66.95554	120.345521	56 11 43
60	19.196491	63.962210	68.41362	121.162801	55 37 21
61	19.888020	65.178046	69.89186	122.003580	55 2 58
62	20.594400	66.402358	71.39084	122.868440	54 28 35
63	21.315910	67.635500	72.91145	123.758155	53 54 13
64	22.052701	68.877606	74.45432	124.673361	53 19 50
65	22.805074	70.129059	76.02042	125.614906	52 45 27
66	23.573186	71.389994	77.61043	126.583487	52 11 5
67	24.357371	72.660825	79.22540	127.580036	51 36 42
68	25.157787	73.941697	80.86608	128.605306	51 2 19
69	25.974778	75.233031	82.53360	129.660301	50 27 57
70	26.808551	76.535188	84.22878	130.745895	49 53 34
71	27.659459	77.848058	85.95285	131.863168	49 19 11
72	28.527710	79.172384	87.70674	133.013056	48 44 49
73	29.413697	80.508436	89.49175	134.196771	48 10 26
74	30.317647	81.856432	91.30890	135.415343	47 36 4
75	31.239989	83.216866	93.15964	136.670112	47 1 41
76	32.180961	84.589966	95.04510	137.962209	46 27 18
77	33.140961	85.975963	96.96618	139.293095	45 52 56
78	34.120421	87.375961	98.92611	140.664048	45 18 33
79	35.119618	88.789594	100.92453	142.076604	44 44 10
80	36.139051	90.214639	102.96381	143.532386	44 9 48
81	37.179043	91.660396	105.04542	145.032900	43 35 25
82	38.240111	93.118455	107.17133	146.579992	43 1 2
83	39.322622	94.592159	109.34320	148.175357	42 26 40
84	40.427139	96.082135	111.56319	149.821051	41 52 17
85	41.554052	97.588753	113.82816	151.518952	41 17 54
86	42.703981	99.112699	116.15555	153.271369	40 43 32
87	43.877350	100.654374	118.53239	155.080397	40 9 9
88	45.074822	102.214506	120.96637	156.948608	39 34 46
89	46.296874	103.793554	123.45986	158.878369	39 0 24
90	47.544231	105.392291	126.01578	160.872559	38 26 1
91	48.817411	107.011233	128.63685	162.933851	37 51 39
92	50.117199	108.651210	131.32634	165.065469	37 17 16
93	51.444173	110.312786	134.08729	167.270444	36 42 53
94	52.799201	111.996881	136.92343	169.552431	36 8 31
95	54.182891	113.704104	139.83816	171.914846	35 34 8
96	55.596244	115.435462	142.83573	174.361831	34 59 45
97	57.039914	117.191641	145.92002	176.897299	34 25 23
98	58.514946	118.973717	149.09580	179.525931	33 51 0
99	60.032087	120.782488	152.36759	182.252247	33 16 37
100	61.562647	122.619117	155.74077	185.081573	32 42 15